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Stability of Neutral Singular Differential Systems with Multiple Time-varying Delays*

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Abstract: The stability problem for singular neutral functional differential systems is discussed in this paper. By applying a new V-functional method and the stability of the difference operator, some asymptotic stability criteria are derived for neutral singular differential systems with multiple time-varying delays. The criteria are described as matrix equations or matrix inequalities, which are flexible and efficient in numerical experiments. Some examples are given to illustrate the results.

Keywords: singular differential systems; V-functional; difference operator; asymptotic stability

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1 Introduction

In this paper, we consider neutral singular differential systems with multiple time-varying delays as follows

$$E\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + \sum_{i=1}^{m} B_i x\big(t-\tau_i(t)\big) + \sum_{i=1}^{m} F_i\big(x(t-\tau_i(t))\big) + F_0\big(x(t)\big), \qquad (1)$$

where $t \geq 0$, $E, C, A, B_i \in \mathbf{R}^{n\times n}$, $\operatorname{rank}(E) = r < n$. The nonlinear part of system $F_j \in \mathbf{R}^n$ is differential, and satisfies $F_j(0) = 0$, $j = 0, 1, \cdots, m$. $\tau_i(t)$ is differential and $0 \leq \tau_i(t) \leq \tau$, $\dot{\tau}_i(t) \leq d_i < 1$ and $t_0 = \inf\limits_{t\geq 0, i=1,\cdots,m}(t-\tau_i(t))$. Suppose $F_j(\cdot)\,(j = 0, 1, \cdots, m)$ is the higher order term in $(\cdot)$, that is

$$\lim_{\|x\|\to 0} \frac{\|F_j(x)\|}{\|x\|} = 0, \quad j = 0, 1, \cdots, m. \qquad (2)$$

Recently, the stability analysis of time-delay systems receives much attention of researchers. Many excellent results for stability of the systems with delay have been obtained[1-10].

Singular differential equations with delay play important roles in mathematical modeling of real-life problems arising in a wide range of applications, for example, multibody mechanics, prescribed path control, electrical design, chemically reacting systems, biology and

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biomedicine[11,12]. But few studies on stability of singular differential equations with delay have been conducted so far.

The purpose of this paper is to establish the asymptotic stability criteria for system (1). The complex nature of singular neutral functional differential systems causes many difficulties in the analytical treatment of such systems. We introduce the stability of the operator $D(x_t) = Ex(t) - Cx(t-\tau)$ and investigate V-functional method of the general singular differential delay system to derive the asymptotic stability criteria. The stability results have been described as matrix equations or matrix inequalities which are computationally flexible and efficient.

2 Stability of the difference operator

In this section, we introduce the stability of the difference operator.

To simplify the discussion, the operator $D : C([-\tau, 0], \mathbf{R}^n) \to \mathbf{R}^n$ is defined by

$$D(x_t) = Ex(t) - Cx(t-\tau). \tag{3}$$

Thus system (2) can be written as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}D(x_t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-\tau_i(t)) + \sum_{i=1}^{m} F_i\big(x(t-\tau_i(t))\big) + F_0\big(x(t)\big). \tag{4}$$

Definition 2.1[10] The operator D is said to be stable if the zero solution of the homogeneous difference equation

$$D(x_t) = 0, \quad t \geq 0, \quad x_0 = \varphi \in \big\{\psi \in C([-\tau, 0], \mathbf{R}^n) : D\psi = 0\big\} \tag{5}$$

is uniformly asymptotically stable.

Lemma 2.1[10] The operator $\widetilde{D} : C([-\tau, 0], \mathbf{R}^n) \to \mathbf{R}^n$ is defined to be

$$\widetilde{D}(x_t) = x(t) - Cx(t-\tau),$$

if $\|C\| < 1$, then the operator $\widetilde{D}$ is stable, where $\|\cdot\|$ is any matrix norm.

Since $\mathrm{rank}(E) = r < n$, then it is easy to find there exist nonsingular constant matrices P_1 and Q_1, such that

$$P_1 E Q_1 = \begin{bmatrix} \overline{E}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_1 C Q_1 = \begin{bmatrix} \overline{C}_1 & \overline{D}_1 \\ \overline{D}_2 & \overline{C}_2 \end{bmatrix}.$$

When $\det\overline{C}_2 \neq 0$, there exist nonsingular matrices P and Q such that

$$PEQ = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad PCQ = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \tag{6}$$

where $C_2 = \overline{C}_1 - \overline{D}_1 \overline{C}_2^{-1} \overline{D}_2$, E_1 is an $r \times r$ nonsingular matrix, C_1 and C_2 are $r \times r$ and $(n-r) \times (n-r)$ constant matrices.

Lemma 2.2[6] The operator D is stable if $\|E_1^{-1} C_1\| < 1$ and $\det C_2 \neq 0$.

3 V-functional method for singular FDEs

In this section, we will introduce the V-functional method for general singular functional differential equations.

Consider the general singular functional differential equation (FDE) as follows

$$E\dot{x}(t) = f(t, x_t), \tag{7}$$

where $f(t,\varphi) : \mathbf{R} \times \bar{C} \to \mathbf{R}^n$, $\bar{C} = C([-\tau,0],\mathbf{R}^n)$, and $f(0,0) = 0$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $E \in \mathbf{R}^{n\times n}$ and $\mathrm{rank}(E) < n$. For $\varphi \in \bar{C}$, we define

$$\|\varphi\| = \sup_{\theta\in[-\tau,0]} |\varphi(\theta)|. \tag{8}$$

In this paper, we always suppose that the conditions to the existence of the solutions for system (7) are satisfied.

Lemma 3.1 For system (7), if there exists a definite V-functional $V(\varphi)$, $\varphi \in \bar{C}$, such that $\dot{V}(\varphi)$ is a semi-definitive functional with the symbol that opposite to the symbol of $V(\varphi)$, or $\dot{V}(\varphi)$ identically equals to zero, then the zero solution of system (7) is stable.

Lemma 3.2 For systems (7), if there exists a definite V-functional $V(\varphi)$, $\varphi \in C$, such that $\dot{V}(\varphi)$ is a definite functional with the symbol opposite to the symbol of $V(\varphi)$, then the zero solution of system (7) is asymptotically stable.

Lemma 3.3 For any x, $y \in R^n$, $\gamma > 0$, inequality $2x^Ty \le \gamma x^Tx + \frac{1}{\gamma}y^Ty$ holds.

4 Asymptotic stability criteria for singular NFDE

In this section, some asymptotic stability criteria for system (1) are derived by applying the V-functional method and stability of difference operator.

The necessary condition for the stability of system (1) is that the operator D be stable[10]. The following theorem governs the system (1).

Theorem 4.1 Suppose the operator D is stable and B_i $(i = 1, 2, \cdots, m)$ are nonsingular. The zero solution of system (1) is asymptotically stable, if there exists a symmetric and positive-definite matrix P such that the following matrix equation

$$A^TPE + E^TPA + \sum_{i=1}^{m}\Big(\varepsilon B_i^TPB_i + \frac{1}{\varepsilon(1-d_i)}E^TPE\Big) + E^TQE = 0 \tag{9}$$

holds, where ε is a given positive number and Q is a given positive-definite matrix.

Proof Construct V-functional as follows

$$\begin{aligned} &V(t, x_t) = V_1(t, x_t) + V_2(t, x_t),\\ &V_1(t, x_t) = (Ex(t))^TPEx(t),\\ &V_2(t, x_t) = \varepsilon \sum_{i=1}^{m} \int_{t-\tau_i(t)}^{t} x^T(s)B_i^TPB_ix(s)\mathrm{d}s. \end{aligned} \tag{10}$$

Since B_i are nonsingular and $P > 0$, we know $B_i^T PB_i > 0$. Thus $V(t, x_t)$ is positive-definite. A straight-forward computation gives the time derivative of the V-functional as

$$\dot{V}_1(t,x_t) = x^T(t)(A^TPE + E^TPA)x(t) + 2x^T(t)E^TPF_0\big(t,x(t)\big)$$

$$+2\sum_{i=1}^{m} x^T(t)E^TPB_ix\big(t-\tau_i(t)\big) + 2\sum_{i=1}^{m} x^T(t)E^TPF_i\big(t,x(t-\tau_i(t))\big),$$

$$\dot{V}_2(t,x_t) = \varepsilon\sum_{i=1}^{m} x^T(t)B_i^TPB_ix(t) - \varepsilon\sum_{i=1}^{m} x^T\big(t-\tau_i(t)\big)B_i^TPB_ix\big(t-\tau_i(t)\big)\big(1-\dot{\tau}_i(t)\big).$$

Since P is a positive-definite symmetric matrix, it has a decomposition $P = U^TU$. From Lemma 3.3, we get

$$2x^T(t)E^TPB_ix\big(t-\tau_i(t)\big) = 2x^T(t)E^TU^TUB_ix\big(t-\tau_i(t)\big)$$

$$\leq \frac{1}{\alpha\varepsilon(1-d_i)}x^T(t)E^TU^TUx(t) + \alpha\varepsilon(1-d_i)x^T\big(t-\tau_i(t)\big)B_i^TU^TUB_ix\big(t-\tau_i(t)\big)$$

$$= \frac{1}{\alpha\varepsilon(1-d_i)}x^T(t)E^TPEx(t) + \alpha\varepsilon(1-d_i)x^T\big(t-\tau_i(t)\big)B_i^TPB_ix\big(t-\tau_i(t)\big),$$

$$2x^T(t)E^TPF_0\big(t,x(t)\big) \leq \frac{1}{\beta}x^T(t)E^TP^2Ex(t) + \beta F_0^T\big(t,x(t)\big)F_0\big(t,x(t)\big),$$

$$2x^T(t)E^TPF_i\big(t,x(t-\tau_i(t))\big) \leq \gamma_iF_i^T\big(t,x(t-\tau_i(t))\big)F_i\big(t,x(t-\tau_i(t))\big) + \frac{1}{\gamma_i}x^T(t)E^TP^2Ex(t),$$

where α, β, $\gamma_i\,(i = 1, 2, \cdots, m)$ are positive numbers. From above, it can be obtained

$$\dot{V}(t,x_t) \leq x^T(t)\Big[-Q + \sum_{i=1}^{m}\frac{1}{\varepsilon(1-d_i)}\Big(\frac{1}{\alpha}-1\Big)E^TPE + \Big(\frac{1}{\beta}+\sum_{i=1}^{m}\frac{1}{\gamma_i}\Big)E^TP^2E\Big]x(t)$$

$$+\sum_{i=1}^{m}(\alpha-1)\varepsilon(1-d_i)x^T\big(t-\tau_i(t)\big)B_i^TPB_ix\big(t-\tau_i(t)\big)$$

$$+\sum_{i=1}^{m}\gamma_iF_i^T\big(t,x(t-\tau_i(t))\big)F_i\big(t,x(t-\tau_i(t))\big) + \beta F_0^T\big(t,x(t)\big)F_0\big(t,x(t)\big).$$

We can choose candidate positive number $\alpha(<1)$, β, $\gamma_i\,(i = 1, 2, \cdots, m)$ with $1-\alpha$ sufficient small and β, γ_i sufficiently large, such that

$$N_0 = -Q + \sum_{i=1}^{m}\frac{1}{\varepsilon(1-d_i)}\Big(\frac{1}{\alpha}-1\Big)E^TPE + \Big(\frac{1}{\beta}+\sum_{i=1}^{m}\frac{1}{\gamma_i}\Big)E^TP^2E < 0. \tag{11}$$

Since $0 < \alpha < 1$ and $B_i\,(i = 1, 2, \cdots, m)$ are nonsingular, we obtain

$$N_i = (\alpha-1)\varepsilon(1-d_i)B_i^TPB_i < 0, \quad i = 1, 2, \cdots, m. \tag{12}$$

Then we have

$$\dot{V}(t,x_t) \leq x^T(t)N_0x(t) + \sum_{i=1}^{m}x^T\big(t-\tau_i(t)\big)N_ix\big(t-\tau_i(t)\big)$$

$$+\sum_{i=1}^{m}\gamma_iF_i^T\big(x(t-\tau_i(t))\big)F_i\big(x(t-\tau_i(t))\big) + \beta F_0^T\big(x(t)\big)F_0\big(x(t)\big).$$

From (11) and (12), we can choose $\sigma > 0$, such that

$$N_j + \sigma I < 0, \quad j = 0, 1, 2, \cdots, m. \tag{13}$$

From (2), there exists $\delta > 0$, when $\|x(t)\| < \delta,\ t \geq t_0$, the following inequalities hold simultaneously

$$\begin{aligned} &\big\|F_0(t, x(t))\big\|^2 \leq \frac{\sigma}{\beta}\big\|x(t)\big\|^2, \\ &\big\|F_i(t, x(t-\tau_i(t)))\big\|^2 \leq \frac{\sigma}{\gamma_i}\big\|x(t-\tau_i(t))\big\|^2. \end{aligned} \tag{14}$$

Then we obtain

$$\dot{V}(t, x_t) \leq x^T(t)(N_0 + \sigma I)x(t) + \sum_{i=1}^{m} x^T\big(t-\tau_i(t)\big)(N_i + \sigma I)x\big(t-\tau_i(t)\big).$$

From (13) we know $\dot{V}(t, x_t)$ is negative definite. Since the operator D is stable, according to Lemma 3.2, we know that the zero solution of system (1) is asymptotically stable and this completes the proof.

Theorem 4.2 Suppose the operator D is stable, the zero solution of system (1) is asymptotically stable, if there exists symmetric and positive-definite matrices P and Q such that the following matrix equation and inequality hold simultaneously

$$A^T PE + E^T PA = -(m+1)E^T QE, \tag{15}$$

$$\sum_{i=1}^{m} \frac{1}{\sqrt{1-d_i}}\|E^T PB_i\| < \lambda_{\min}(Q). \tag{16}$$

Proof Similarly to the proof of Theorem 4.1, the result follows from the V-functional method and stability of difference operator.

If $m = 1$ and $F_j \equiv 0\,(j = 0, 1)$ hold, system (1) reduces to

$$E\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx\big(t-\tau(t)\big). \tag{17}$$

Furthermore, if $t - \tau(t) = qt\,(0 < q < 1$ is constant) holds, system (17) is written as

$$E\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx(qt). \tag{18}$$

From Theorem 4.1 and Theorem 4.2, we have:

Corollary 4.1 Suppose the operator D is stable, the zero solution of system (17) is asymptotically stable, if there exists matrices $P > 0,\ Q > 0$, and a positive parameter ε such that one of the following two conditions is satisfied

$$(\mathrm{H}_1): \quad A^T PE + E^T PA + \varepsilon B^T PB + \frac{1}{\varepsilon(1-d)}E^T PE + E^T QE = 0,$$

$$(\mathrm{H}_2): \quad \begin{cases} A^T PE + E^T PA = -2E^T QE, \\ \|E^T PB\| < \sqrt{1-d}\,\lambda_{\min}(Q). \end{cases}$$

Corollary 4.2 Suppose the operator D is stable, the zero solution of system (17) is asymptotically stable, if there exists matrices $P > 0,\ Q > 0$, and a positive parameter ε such that one of the following two conditions is satisfied:

$$(\mathrm{H}_3): \quad A^TPE + E^TPA + \varepsilon B^TPB + \frac{1}{\varepsilon q}E^TPE + E^TQE = 0,$$

$$(\mathrm{H}_4): \quad \begin{cases} A^TPE + E^TPA = -2E^TQE, \\ \|E^TPB\| < \sqrt{q}\,\lambda_{\min}(Q). \end{cases}$$

5 Illustrative examples

In this section, we give some examples to illustrate the derived stability results.

Example 5.1 Consider the following second-order singular differential system with time-varying delay

$$\begin{cases} \dot{x_1}(t) - \frac{1}{2}\dot{x_1}\big(t-\tau(t)\big) = -x_1(t) + \frac{1}{2}x_1\big(t-\tau(t)\big) + x_1^2(t)x_2(t) + x_1^6(t), \\ -\dot{x_2}\big(t-\tau(t)\big) = x_2(t) + x_1^4(t). \end{cases} \tag{19}$$

First, since $E_1 = 1,\ C_1 = \frac{1}{2},\ C_2 = 1$, it is easy to know that $\|E_1^{-1}C_1\| < 1$ and $\det C_2 \neq 0$. From Lemma 2.2, we know the operator D is stable. Secondly, since $\tau(t) = 0.4t,\ \dot{\tau}(t) = 0.4$, the matrices P and Q and the positive number d in the condition (H_2) of Corollary 1 are chosen as follows

$$d = 0.4, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then we obtain

$$A^TPE + E^TPA = -2E^TQE, \quad \|E^TPB\| < \sqrt{1-d}\,\lambda_{\min}(Q).$$

Thus, the condition (H_2) of Corollary 1 is satisfied. The zero solution of system (19) is asymptotically stable.

Example 5.2 Consider the following third-order singular differential system with time-varying delay

$$\begin{cases} \dot{x}_1(t) - \frac{1}{2}\dot{x_1}\big(t-\tau(t)\big) = -x_1(t) + x_2(t) + x_1\big(t-\tau(t)\big) + x_2\big(t-\tau(t)\big) + \sin\big(x_1(t-\tau(t))\big), \\ \dot{x}_2(t) - \frac{1}{2}\dot{x_2}\big(t-\tau(t)\big) = -2x_1(t) - 4x_2(t) + x_2\big(t-\tau(t)\big) + \sin\big[x_1(t-\tau(t)) + x_2(t-\tau(t))\big], \\ x_3(t) - \dot{x_3}\big(t-\tau(t)\big) = x_1(t) + x_2(t) - x_2\big(t-\tau(t)\big) + \sin\big(x_1(t)\big) + \sin\big(x_3(t-\tau(t))\big). \end{cases} \tag{20}$$

First, since

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad C_2 = 1,$$

it is easy to know that $\|E_1^{-1}C_1\| < 1$ and $\det C_2 \neq 0$. From Lemma 2.2, we know that the operator D is stable. Secondly, since

$$\tau(t) = \mathrm{e}^{-t} + \frac{t}{2} - \frac{1}{2}, \quad \dot{\tau}(t) = -\mathrm{e}^{-t} + \frac{1}{2} < \frac{1}{2},$$

the matrices P, Q and the positive number ε, d in the condition (H_1) of Corollary 1 are chosen as follows

$$d=\frac{1}{2},\quad \varepsilon=1,\quad P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad Q=\begin{bmatrix}1&0&0\\0&3&0\\0&0&2\end{bmatrix}>0.$$

Then we obtain

$$A^TPE+E^TPA+\varepsilon B^TPB+\frac{1}{\varepsilon(1-d)}E^TPE+E^TQE=0.$$

Thus, the condition (H_1) of Corollary 1 is satisfied. The zero solution of system (20) is asymptotically stable.

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具多变时滞中立型奇异微分系统的稳定性

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摘 要: 本文讨论了中立型奇异泛函微分系统的稳定性问题. 利用V泛函方法和差分算子的稳定性获得具变时滞中立型奇异微分系统的渐近稳定性判据. 所得结果被描述为矩阵等式或者矩阵不等式，在计算上是可行和有效的，并给出例子说明了所得结果.

关键词: 奇异微分系统; V泛函; 差分算子; 渐近稳定性